15.4 Center of Mass

New App: A thin plate (a *lamina*) with density at each point given by

$$\rho(x, y) = \text{mass/area} \quad (\text{kg/m}^2),$$

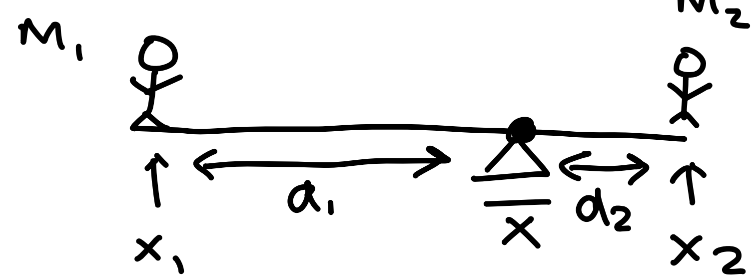
we will see that the center of mass (centroid) is given by

$$\bar{x} = \frac{\text{Moment about y}}{\text{Total Mass}} = \frac{\iint_R x\, p(x,y)\,dA}{\iint_R \; p(x,y)\,dA}$$

$$\bar{y} = \frac{\text{Moment about x}}{\text{Total Mass}} = \frac{\iint_R y\, p(x,y)\,dA}{\iint_R \; p(x,y)\,dA}$$

total mass

Motivation "the see-saw"



$$M_1 d_1 = M_2 d_2$$

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

In general: If you are given ***n* points** $(x_1,y_1), (x_2,y_2), \ldots, (x_n,y_n)$ with corresponding masses $m_1, m_2, \ldots, m_n$ then

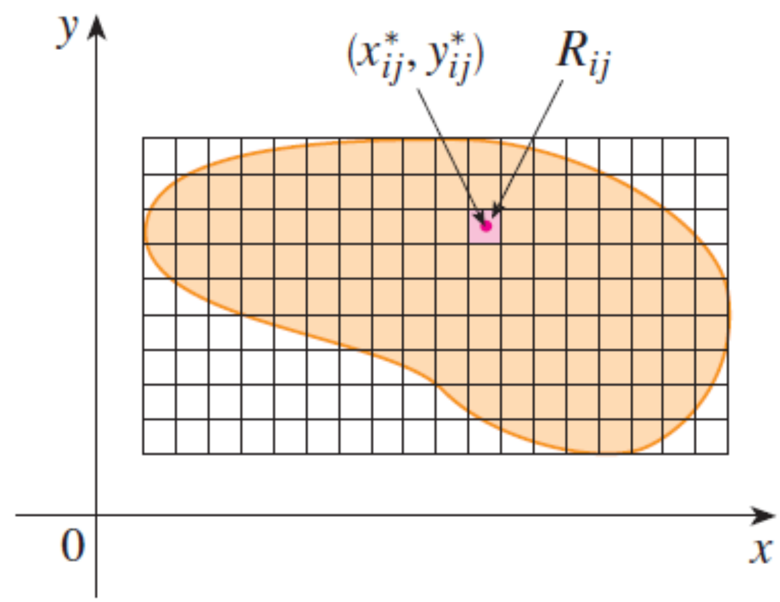
$$\bar{x} = \frac{m_1x_1 + \cdots + m_nx_n}{m_1 + \cdots + m_n} = \frac{M_y}{M}$$

$$\bar{y} = \frac{m_1y_1 + \cdots + m_ny_n}{m_1 + \cdots + m_n} = \frac{M_x}{M}$$

Derivation:

1. Break region into m rows and n columns.
2. Find center of mass of each rectangle:
$$(\bar{x}_{ij}, \bar{y}_{ij})$$
3. Estimate the mass of each rectangle:
$$m_{ij} = p(\bar{x}_{ij}, \bar{y}_{ij})\Delta A$$
4. Now use the formula for *n* points.
5. Take the limit.

$$\bar{x} = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}} = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} \bar{x}_{ij}\, p(\bar{x}_{ij}, \bar{y}_{ij})\Delta A}{\sum_{i=1}^{m}\sum_{j=1}^{n} p(\bar{x}_{ij}, \bar{y}_{ij})\Delta A}$$



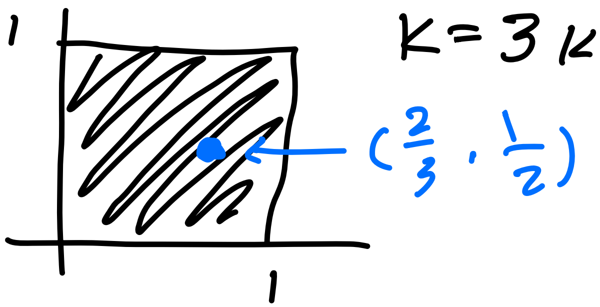
Center of Mass:

$$\bar{x} = \frac{\text{Moment about y}}{\text{Total Mass}} = \frac{\iint_R x\, p(x,y)\,dA}{\iint_R p(x,y)\,dA}$$

$$\bar{y} = \frac{\text{Moment about x}}{\text{Total Mass}} = \frac{\iint_R y\, p(x,y)\,dA}{\iint_R p(x,y)\,dA}$$

Example:
Consider a 1 by 1 m square metal plate. The density is given by *p(x,y) = kx* kg/m^2 for some constant *k.*
Find the center of mass.

Side note:
The density *p(x,y) = kx* means that the density is *proportional* to *x* which can be thought of as the distance from the *y*-axis.
In other words, the plate gets heavier at a constant rate from left-to-right.



$K = 3 \text{ kg/m}^2$

$P(x,y) = Kx$

$x=1 \Rightarrow P(1,0) = \overset{3}{K} \cdot 1$

$x=0 \Rightarrow P(0,0) = K \cdot 0$

$$\bar{x} = \frac{\iint_R x \cdot p(x,y)\, dA}{\iint_R p(x,y)\, dA} = \frac{\int_0^1 \int_0^1 x \cancel{k} x \, dx\, dy}{\int_0^1 \int_0^1 \cancel{k} x \, dx\, dy}$$

$$\bar{x} = \frac{\int_0^1 \frac{1}{3} x^3 \Big|_0^1 \, dy}{\int_0^1 \frac{1}{2} x^2 \Big|_0^1 \, dy} = \frac{\frac{1}{3}}{\frac{1}{2}} = \boxed{\frac{2}{3}}$$

$$\boxed{\bar{y} = \frac{1}{2}}$$

Translations: getting heavier...

Density proportional to the dist. from…

…the y-axis -- $p(x, y) = kx$.

…the x-axis -- $p(x, y) = ky$.

…the origin -- $p(x, y) = k\sqrt{x^2 + y^2}$.

Density proportional to the square of the distance from the origin:

$$p(x, y) = k(x^2 + y^2).$$

Density inversely proportional to the distance from the origin:

$$p(x, y) = \frac{k}{\sqrt{x^2 + y^2}}$$

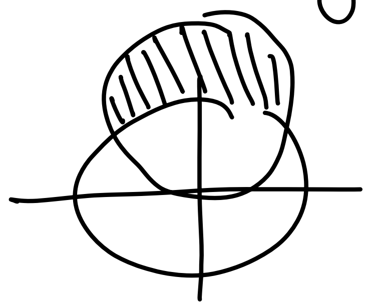
getting lighter

Example: A thin plate is in the shape of the region bounded between the circles of radius 1 and 2 in the first quadrant. The density is proportional to the distance from the origin. Find the center of mass.

getting heavier as you get farther away

6 on HW

Find the region



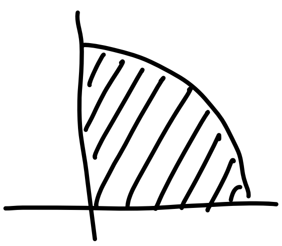
$$p(x,y) = \frac{k}{\sqrt{x^2+y^2}}$$

$$\frac{\iint_R x\, p(x,y)\, dA}{\iint_R p(x,y)\, dA}$$

$\bar{x} = 0$?

5 on HW

$p(x,y) = ky$



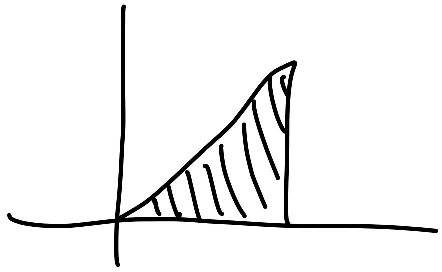
$$\frac{\iint_R x\, p(x,y)\, dA}{\iint_R p(x,y)\, dA} = \bar{x}$$

$(\bar{x}, \bar{y})$

$$\frac{\iint_R y\, p(x,y)\, dA}{\iint_R p(x,y)\, dA} = \bar{y}$$

4 on HW

$$\frac{1}{\text{Area}} \iint_R 3xy \, dA$$



1-3 on HW

draw region → think in terms of picture

1 = True

2 = false

3 ⇒ compute and see if same ⇒ True